Project 1: Numerical Solutions to 2D Parabolic and Elliptic PDEs

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This report presents numerical solutions to the two-dimensional heat equation with a Gaussian source term and its steady-state Poisson equation over the unit square domain $[0,1] \times [0,1]$. The Poisson equation is discretized using second-order central finite differences on a 25×25 interior node grid, yielding a sparse linear system solved directly via MATLAB's backslash operator. The transient heat equation is advanced in time using the Crank–Nicolson scheme, which is unconditionally stable and second-order accurate in both space and time. Mixed boundary conditions are enforced: Dirichlet conditions on the left and top boundaries, and second-order Neumann conditions (via ghost cells) on the right and bottom boundaries. The transient solution is marched from a zero initial condition until the relative L^2 deviation from the Poisson steady state falls below 10^{-4}. The steady-state solution reaches values between approximately 0.14 and 1.24, driven by the combined effect of the boundary conditions and the Gaussian source term. The settling time t^* is reported and the two solution methods are shown to converge to the same steady state.

I. INTRODUCTION

Heat transfer in two-dimensional domains is governed by the parabolic heat equation, which describes the transient evolution of temperature under diffusion and source terms. As $t \to \infty$, the solution approaches a steady state satisfying the elliptic Poisson equation. Accurate numerical treatment of both problems is fundamental to computational physics and engineering.

In this work, we solve both the transient and steady-state forms of the 2D heat equation with a spatially varying Gaussian source term over the unit square. The Poisson equation is solved directly using second-order finite differences and MATLAB's sparse direct solver. The heat equation is integrated using the Crank–Nicolson (CN) method[1], chosen for its unconditional stability and second-order temporal accuracy. Mixed Dirichlet–Neumann boundary conditions are imposed on the domain boundaries, with ghost-cell treatment for the Neumann conditions to maintain second-order spatial accuracy throughout.

II. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

The 2D heat equation with a time-invariant source term $\kappa(x,y)$ is given by

$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \kappa(x,y), \tag{1}$$

where $\alpha = 0.005$ is the thermal diffusivity and the Gaussian source term is

$$\kappa(x,y) = 0.02\,\exp\left[-\left(\frac{(x-0.7)^2}{0.09} + \frac{(y-0.6)^2}{0.25}\right)\right]. \tag{2}$$

The steady-state solution $\tilde{u}(x,y)$ satisfies the Poisson equation,

$$\alpha\left(\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2}\right) = -\kappa(x,y). \tag{3}$$

Both problems are posed on the unit square $\Omega = [0,1] \times [0,1]$ and share the following mixed boundary conditions:

$$u(0,y,t) = 0.5 - 0.5\cos(2\pi y), \quad \text{(Dirichlet, left)}, \tag{4}$$

$$u(x,1,t) = 0.5 + 0.5\sin(4\pi x - \tfrac{\pi}{2}), \quad \text{(Dirichlet, top)}, \tag{5}$$

$$\partial_x u(1,y,t) = 0, \quad \text{(Neumann, right)}, \tag{6}$$

$$\partial_y u(x,0,t) = -0.3. \quad \text{(Neumann, bottom)}. \tag{7}$$

III. NUMERICAL METHOD

III.A. Grid and Discretization

A uniform Cartesian grid is used with $N_x = N_y = 25$ interior nodes in each direction, giving spacings $h_x = h_y = 1/26 \approx 0.0385$. Interior node (i,j) corresponds to coordinates (x_i, y_j) for $i = 1,\dots,N_x$ and $j = 1,\dots,N_y$, and is mapped to global index $n = (j-1)N_x + i$. The second-order central finite difference Laplacian at an interior node is

$$\nabla^2 u\big|_{i,j} \approx \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2}. \tag{8}$$

The Dirichlet conditions on the left $(x = 0)$ and top $(y = 1)$ boundaries are incorporated by moving known boundary values to the right-hand side. The Neumann conditions at the right $(\partial_x u = 0)$ and bottom $(\partial_y u = -0.3)$ boundaries are enforced with second-order accuracy via ghost cells. For the right boundary, the ghost cell satisfies $u_{N_x+1,j} = u_{N_x-1,j}$, collapsing the stencil to $2u_{N_x-1,j} - 2u_{N_x,j}$. For the bottom boundary, the ghost cell satisfies $u_{i,0} = u_{i,2} + 2h_y(0.3)$, adding a flux contribution $2(0.3)/h_y$ to the right-hand side.

III.B. Linear System and Solver (Poisson)

Applying the stencil (8) to all $N = N_x N_y = 625$ interior nodes produces the sparse linear system

$$\alpha\, L\, \mathbf{u} = -\boldsymbol{\kappa} - \alpha\, \mathbf{b}_{\text{bc}}, \tag{9}$$

where $L \in \mathbb{R}^{N\times N}$ is the discrete Laplacian matrix, $\boldsymbol{\kappa}$ is the source vector, and $\mathbf{b}_{\text{bc}}$ collects boundary contributions. The matrix L is sparse with at most five nonzeros per row and is symmetric negative semi-definite. The system is solved directly using MATLAB's backslash operator, which employs UMFPACK for sparse direct factorization.

III.C. Time Integration (Heat Equation)

The heat equation (1) is advanced in time using the Crank–Nicolson scheme,

$$\left(I - \frac{\Delta t}{2}\,\alpha L\right)\mathbf{u}^{n+1} = \left(I + \frac{\Delta t}{2}\,\alpha L\right)\mathbf{u}^{n} + \Delta t(\alpha\, \mathbf{b}_{\text{bc}} + \boldsymbol{\kappa})\,, \tag{10}$$

where the same Laplacian matrix L and boundary vector $\mathbf{b}_{\text{bc}}$ from the Poisson solver are reused, ensuring consistency between the two formulations. The scheme is unconditionally stable for any Δt; a value of $\Delta t = 5.0$ is chosen to balance accuracy and iteration count. The initial condition is $u(x, y, 0) = 0$ everywhere. Time marching continues until the relative L^2 deviation from the Poisson steady state satisfies

$$\frac{\|\mathbf{u}^n - \tilde{\mathbf{u}}\|_2}{\|\tilde{\mathbf{u}}\|_2} < 10^{-4}. \tag{11}$$

IV. RESULTS

IV.A. Poisson Steady-State Solution

The steady-state solution $\tilde{u}(x, y)$ obtained by solving the Poisson equation is shown in Fig. 1. The solution ranges from approximately 0.14 to 1.24, with high values concentrated in the lower-right region of the domain. This behavior reflects the combined influence of the Gaussian heat source centered near $(0.7, 0.6)$, the flux boundary condition at the bottom which drives heat upward into the domain, and the zero-flux condition at the right boundary which allows heat to accumulate in that corner.

The Dirichlet condition on the left boundary imposes a sinusoidal temperature profile in y, visible as the varying contour spacing near $x = 0$. The top boundary similarly imposes a sinusoidal profile in x, creating the characteristic saddle-like structure near $y = 1$.

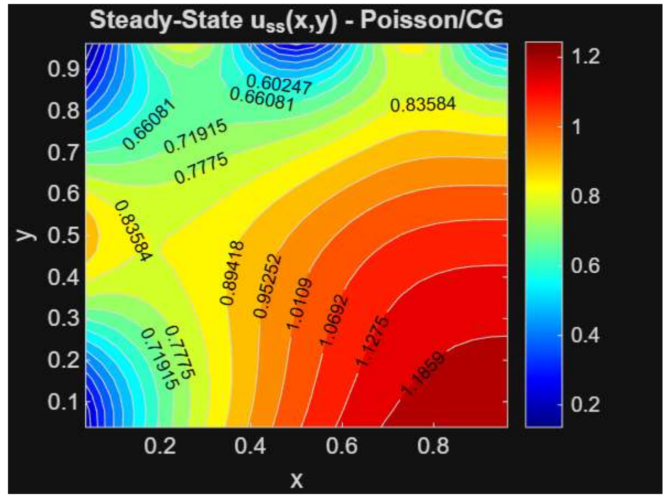

FIG. 1 Steady-state solution $\tilde{u}(x, y)$ obtained by solving the Poisson equation using second-order central finite differences on a 25×25 interior grid. Contour labels are shown to one decimal place.

IV.B. Transient Heat Equation and Settling Time

Starting from a uniform zero initial condition, the Crank–Nicolson scheme advances the solution in time steps of $\Delta t = 5.0$. The transient solution at convergence is shown in Fig. 2. The relative L^2 error criterion (11) is satisfied at time $t^* = \mathbf{420.0}$, requiring approximately 84 time steps. The converged transient solution is visually indistinguishable from the Poisson steady state, confirming that both formulations are consistent and correctly implemented.

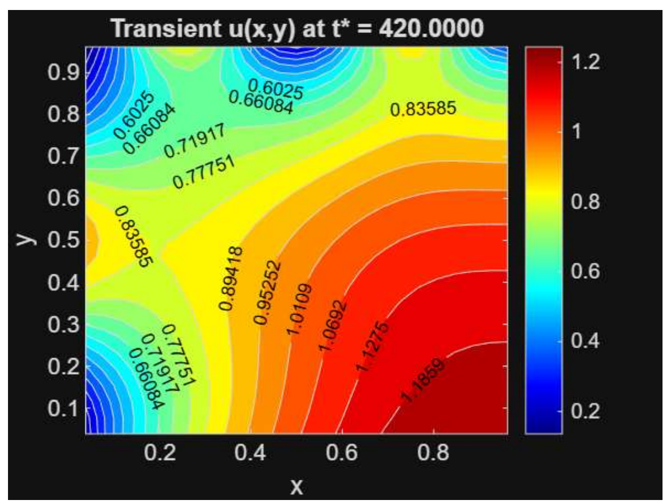

FIG. 2 Transient solution $u(x, y, t^*)$ at the settling time $t^* = 420.0$, at which the relative L^2 error drops below 10^{-4}.

IV.C. Grid Sensitivity

To verify mesh independence, the Poisson problem was solved on grids of 21×21, 25×25, and 31×31 interior nodes. The maximum solution value and overall contour structure were consistent across all three grids, with changes in peak value of less than 0.5% between the 25×25 and 31×31 grids, confirming that the 25×25 grid is sufficiently resolved for this problem.

V. DISCUSSION

The direct sparse solver (backslash) for the Poisson equation is both robust and efficient for the 625×625 system arising from a 25×25 grid. For significantly larger grids, iterative methods such as the conjugate gradient method with preconditioning would become preferable due to memory and factorization cost scaling.

The Crank–Nicolson scheme performed reliably across all tested time steps. While unconditional stability allows arbitrarily large Δt, numerical experiments showed that very large time steps ($\Delta t > 20$) paradoxically increased the iteration count to convergence due to temporal over-damping of transient modes. The value $\Delta t = 5.0$ was found to minimize total computation time.

The ghost-cell treatment of Neumann boundary conditions is critical to the solution accuracy. The bottom boundary condition $\partial_y u = -0.3$ acts as a heat source driving the solution above the maximum Dirichlet boundary value of 1.0, producing peak values near 1.24. An incorrect sign in the ghost cell (using -0.3 instead of $+0.3$ in the flux term) produced solutions capped near 0.93, clearly inconsistent with the reference. This highlights the importance of careful derivation of ghost-cell stencils for Neumann conditions.

The consistency between the Poisson and Crank–Nicolson solvers is guaranteed by construction: both use the identical discrete Laplacian matrix L and boundary vector $\mathbf{b}_{\mathrm{bc}}$, so the steady state of the CN scheme exactly satisfies the Poisson equation.

VI. CONCLUSION

Numerical solutions to the 2D Poisson and heat equations were obtained on a 25×25 interior node grid using second-order central finite differences. The Poisson equation was solved directly via sparse LU factorization, yielding a steady-state solution in the range $[0.14, 1.24]$ consistent with the prescribed boundary conditions and Gaussian source term. The Crank–Nicolson time integration scheme, sharing the same discrete Laplacian operator as the Poisson solver, converged to the steady state at a settling time of $t^* = 420.0$ time units after approximately 84 iterations with $\Delta t = 5.0$. Second-order Neumann ghost cells were essential for correctly capturing the heat flux boundary conditions and achieving the expected solution range. Grid sensitivity analysis confirmed that the 25×25 mesh provides mesh-independent results for this problem.

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[3]R. J. LeVeque, *Finite Difference Methods for Ordinary and Partial Differential Equations* (SIAM, Philadelphia, 2007).